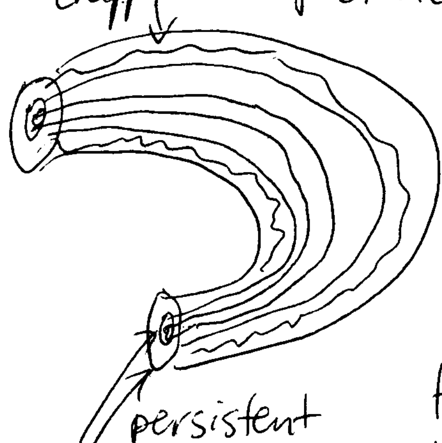
§5. Systems with two degrees of freedom

Thm In an isoenerget. nondegenerate system with two degrees of freedom, for all initial conditions the action variables remain forever near ~~their~~ initial values

Pf (dimensional consideration) $\dim M^{2n} = 4$



energy level $\dim \{H = \text{const}\} = 3$

invar. tori $\dim \mathbb{T}^2 = 2$

2-dim invariant tori divide 3-dim energy level $\{H = \text{const}\}$

Hence a phase curve starting between tori remains there $\Rightarrow I$ is approximately the same. The gap between invar. tori $\sim \sqrt{\varepsilon}$

$$\Rightarrow \quad I - I_0 \sim \sqrt{\varepsilon} \quad \forall t \in \mathbb{R} \quad \blacksquare$$

Ex. For $H(I, \varphi, \varepsilon) = \dfrac{I_1^2 - I_2^2}{2} + \varepsilon \sin(\varphi_1 - \varphi_2)$ the Ham. eq's are

$$\begin{cases} \dot I_1 = -\varepsilon \cos(\varphi_1 - \varphi_2) \\ \dot I_2 = \varepsilon \cos(\varphi_1 - \varphi_2) \\ \dot\varphi_1 = I_1, \quad \dot\varphi_2 = -I_2 \end{cases}$$

There are "fast" solutions: $I_1 = -\varepsilon t, \quad I_2 = \varepsilon t$

$$\varphi_1 = -\varepsilon \frac{t^2}{2}, \quad \varphi_2 = -\varepsilon \frac{t^2}{2}$$

This system is isoenerg. nondegen. $\forall H \neq 0$, but it is not isoen. nondeg for $H = 0$

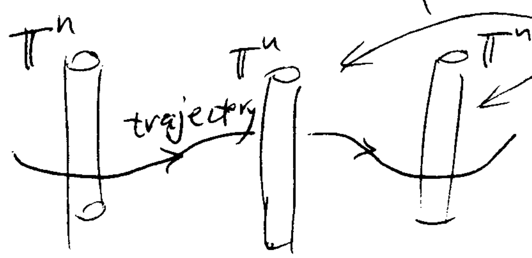
The $\{H = 0\}$-level contains the ray $I_1 = -I_2$, where the frequency ratio $=$ const $= 1$, called "superconductivity chanel".

§6. Diffusion of slow variables in higher dim systems

Now dimension does not prevent the escape of action variables.

If $n \geq 3$, $\dim\{H = const\} = 2n-1 \geq n = \dim \mathbb{T}^n$

codim $\mathbb{T}^n$ in $\{H = const\}^{2n-1}$ is $n-1 \geq 2$



persistent tori inside every level.

<u>Conjecture</u> ("<u>Arnold's diffusion</u>") In a higher-dim problem the typical case is ~~topological~~ instability: for any neighborhood of any phase point there is a phase trajectory along which slow variables go away from the initial values by a quantity of order 1.

J. Mather proved for 3 degrees of freedom for convex Ham's $H(I)$

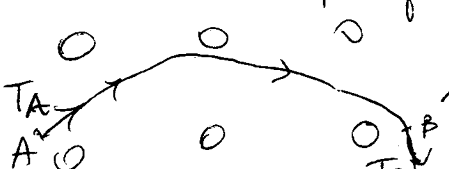
Numerical experiments show: evolution of the action variables is not directional, but looks like a random walk along resonances around invariant tori - "Arnold diffusion"

hence the term

<u>Examples</u> 1) (Kaloshin - Levi 2006)

Newton eq'n in $\mathbb{R}^3$ with periodic potential with bumps at a lattice, freely flying particle:

bump f'n



$$\ddot{X} = -\varepsilon \nabla U(X), \quad U(X) = -\cos^2\frac{\pi z}{2} + \varepsilon^k \beta(X, \varepsilon)$$

$X \in \mathbb{R}^3$ $\exists$ geodesics with U-turns:

$T_A = -T_B$, dist$(A, B) = \bar{o}(1)$ as $t \to \pm\infty$

2) (Arnold ?)

Surface of revolution with small periodic bump

Generically, the Arnold diffusion is exp. slow.

Def (Ilyashenko-Nekhoroshev) An analytic function is steep if it has no critical points and its restriction to any plane of any dim'n has only isolated crit. pts.

Thm (Nekhoroshev 1977) If $H_0(I)$ is a steep function, then $\exists a, b, c > 0$ such that in the perturbed Hamilt. system for a suff. small perturbation we have

$$|I(t) - I(0)| < \varepsilon^b \text{ for } 0 \le t \le \frac{1}{\varepsilon} \exp\left(\frac{1}{c\varepsilon^a}\right), \text{ i.e.}$$

I remains ε^b-close for exp-long time

(Here a, b, c - const's depending on H.)

Pf idea: In nonresonance domains one can remove the perturbation by variable change, move it to exp small terms $\Rightarrow$ the action evolution is exp slow if $\nexists$ resonance

Fast evolution of order ε is possible at a resonance. For steep functions exact resonances occur at isolated pts. The resonance is destroyed during the evolution, i.e. fast evolution goes on for a short time only. $\boxtimes$